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Finite-amplitude thermal convection and geostrophic flow in a rotating magnetic system

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An electrically conducting Boussinesq fluid is confined between two rigid horizontal planes. The fluid is permeated by a strong uniform horizontal magnetic field and the entire system rotates rapidly about a vertical axis. By heating the fluid from below and cooling it from above the system becomes unstable to small perturbations when the adverse temperature gradient becomes sufficiently large. Attention is restricted to small values of the Ekman number E and the ratio q of the thermal and magnetic diffusivities (see (1.2) and (1.3) below). In this parameter range marginal convection is steady and its character depends on the relative sizes of the Coriolis and Lorentz forces as measured by the parameter λ (see (1.1) below). When $\lambda \geqslant 2/3^{\frac{1}{4}}$, motion consists of a single roll, whose axis is perpendicular to the applied magnetic field. On the other hand, when $\lambda < 2/3^{\frac{1}{4}}$, two distinct rolls are possible: the axis of each roll lies oblique but with equal angle to the applied magnetic field. Only the latter case is discussed here.

Once the Rayleigh number R exceeds its critical value R_c only one of the two sets of single rolls remains stable, while its squared amplitude increases linearly with $R-R_c$. For certain values of the parameters λ and τ (see (1.6) below) a second critical value may be isolated at which the system becomes unstable to unidirectional geostrophic flow perturbations aligned with the applied magnetic field. The instability sets in as either a steady or oscillatory shear flow dependent on the values taken by λ and τ. In both cases, after the secondary instability sets in, the roll amplitude remains largely insensitive to further increase in the Rayleigh number with the consequence that the geostrophic flow is stabilized. The amplitude of the shear, on the other hand, increases with R, adjusting its magnitude to ensure stability of the convection rolls.

1. Introduction

Until a few years ago the generally accepted picture of the magnetic field inside the Earth's liquid core was one in which the azimuthal magnetic field is significantly larger than any other component. Since the azimuthal field is not observed at the Earth's surface, there is no direct evidence for this belief. On the other hand, since the Coriolis forces are large, it is conjectured that there are vigorous azimuthal motions tending to align the magnetic field in the same direction. This picture forms the basis of Braginsky's (1964) kinematic model of the geodynamo. An alternative model of the

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geodynamo has been proposed by Busse (1975) in which there is no significant azimuthal flow and the magnetic field is maintained simply by small-scale convective motions. Since all components of the magnetic field are of comparable size, the magnetic field strength in the core is weak and its magnitude is comparable to the surface value. The answer to the question, which of the two models most closely resembles the geodynamo, is not known at the present time. Nevertheless recent arguments in support of strong- and weak-field models are given by Roberts (1978) and Busse (1978*a*, *b*) respectively.

In the present paper attention is focused upon aspects of a problem that arise in the strong field case, namely what determines the magnitude of the large azimuthal flow. One possible answer was provided by Taylor (1963). He suggested that, in the case of a spherical container, the average of the azimuthal component of the Lorentz force taken over a circular cylinder, whose axis is coincident with the rotation axis, should vanish. The reason for the assumption is that, if the average is non-zero, and viscosity is neglected, the geostrophic flow is accelerated rapidly. Taylor regarded this behaviour as purely transient and proposed that the magnetic field would adjust itself on a relatively short time scale in such a way that the so-called Taylor condition is satisfied. Indeed in a steady state he was able to show that with the correct choice of geostrophic flow, which is determined simply as a function of the radial distance from the rotation axis, the ensuing solution of the magnetic-induction equation can lead to a magnetic field satisfying his condition.

It is not always clear, however, that Taylor's condition will be automatically satisfied. One such counter-example is the torsional oscillations in a spherical container discussed by Braginsky (1970), amongst others. Consider for simplicity a weak axisymmetric meridional magnetic field permeating a stationary fluid. Superimposed upon this basic state is the possibility of quasi-steady geostrophic flow varying on a time scale which is long compared with the rotation period. The ensuing Lorentz force fails to satisfy Taylor's condition and so geostrophic cylinders are subject to a restoring force. As a result an Alfvén wave, which is not directly affected by the Coriolis force, ensues in the form of torsional oscillations. Furthermore, when dissipative processes are taken into account the oscillation is damped and eventually decays. Of particular interest here is the effect of Ekman layers (e.g. see Roberts & Soward 1972) as they play a central role in the problems discussed in the following sections. There are, however, cases in which perturbations from a state of motion satisfying Taylor's condition lead to systematic growth of the geostrophic flow (e.g. see Roberts & Stewartson 1975, and §§ 4 to 6 below) and they contrast dramatically with the simple Alfvén wave picture described above. It is exactly this instability which provides the main theme of this paper.

Evidently steady fluid motions together with any related instabilities can be sustained in the presence of dissipation only if there is an energy source. In the case of the geodynamo this is generally supposed to be thermal. A simple model, which isolates the dynamic influence of a strong azimuthal magnetic field on thermal convection in a rotating system, is as follows. An electrically conducting Boussinesq fluid of constant density ρ is confined between two horizontal slippery planes a distance d apart. The fluid is permeated by a strong uniform horizontal magnetic field $\mathbf{B}_0$ and the entire system rotates about a vertical axis with angular velocity $\mathbf{\Omega}$. The upper and lower planes are maintained at constant temperatures $T_0 - \Delta T$ and T_0 respectively.

Both boundaries are assumed to be perfect electrical conductors. The choice of all boundary conditions is motivated by expediency but it is generally believed that more realistic conditions do not significantly alter the conclusions obtained from the model. By contrast, a more realistic geometry may alter the results considerably. This has already been demonstrated by Roberts & Loper (1979), Soward (1979) and Fearn (1979), who have investigated linear problems involving circular magnetic-field lines and have found other magnetically driven instabilities in addition to those of thermal origin.

The thermal instability of rotating magnetic systems has been discussed by a number of authors including Chandrasekhar (1961). A detailed analysis of several different configurations, which include the above model, has been given more recently by Eltayeb (1972). The principal conclusion arrived at by these authors is that whether the onset of instability is dominated by rotational or magnetic effects depends upon the size of

$$\lambda = 2\rho\Omega/\sigma B_0^2, \tag{1.1}$$

where σ is the electrical conductivity. Roberts & Stewartson (1974) have discussed the particular problem outlined above in the case for which λ is of order unity (see (2.2*c*) below) and the Ekman number

$$E = \nu/\Omega d^2, \tag{1.2}$$

where ν is the viscosity, is small (see (2.2*a*) below). In this parameter range marginal convection is steady or oscillatory depending on the values of λ and q, where

$$q = \kappa/\eta \tag{1.3}$$

is the ratio of the thermal and magnetic diffusivities κ and η respectively. Roberts & Stewartson (1974) indicate clearly in their paper (subsequently referred to as RS I) the regions in the q, λ plane where steady convection can occur. For small values of q (see (2.2*b*) below), of particular interest to the geodynamo, marginal convection is always steady.

When the uniform magnetic field in our plane-layer model is not horizontal but vertical (say), then it is well known that the modified Rayleigh number

$$R = g\alpha\Delta T d/2\pi^2\kappa\Omega, \tag{1.4}$$

where g is the acceleration due to gravity and α is the coefficient of expansion, is minimized by order one values of λ. On the other hand, for the special case of a uniform horizontal magnetic field the situation is a little different. Thus, when the magnetic field is weak, $\lambda \geqslant 2/3^{\frac{1}{2}}$, instability is characterized by convection rolls whose axes lie perpendicular to the magnetic field. In this regime the magnetic field is said to relax the rotational constraint and so the critical Rayleigh number R_c decreases with λ. For larger fields, $\lambda < 2/3^{\frac{1}{2}}$ (see also (3.1) below), two sets of convection rolls are possible whose axes make angles of $+\gamma$ and $-\gamma$ respectively with the magnetic field. The angle γ selected ensures that an optimal balance is struck between the relative sizes of the Coriolis and Lorentz forces. Hence as λ decreases from $2/3^{\frac{1}{2}}$ to 0, γ decreases from $\frac{1}{2}\pi$ to 0, while R_c remains constant.

The finite amplitude dynamics of single-roll convection was discussed in RS I. Roberts & Stewartson (1975) also analysed the stability of oblique rolls, which occur when $\lambda < 2/3^{\frac{1}{2}}$, to perturbations of the second roll system. This paper will be referred to as RS II. Whereas Taylor's condition is met by a single-roll system, this is not the case for the two-roll system and so a key ingredient in the evolution of the convection rolls is their coupling through the induced geostrophic flow. All values of q were investigated in RS II but of particular interest here are the results for small q. They are that one set of rolls is always unstable while the other set is only unstable in the limited parameter range given approximately by

$$1{\cdot}0794 < \lambda < 2/3^{\frac{1}{2}} \tag{1.5}$$

(see also (4.14) and the remarks following it).

Within the range (1.5) neither roll system is stable and the subsequent evolution of the system once instability sets in is clearly of interest. The answer to this question was not obtained in RS II where only speculations were made of the outcome of the instability. In order to attempt an answer, the problem is reconsidered here with some modifications. Firstly the slippery boundaries are replaced by rigid boundaries so that the geostrophic flow is damped and in the absence of the Lorentz force decays on the time scale

$$\tau(d^2/\kappa\pi^2) = E^{\frac{1}{2}}(d^2/2\nu) \tag{1.6}$$

(see also (2.5) below), where τ is a new dimensionless parameter defined by (1.6). Secondly attention is restricted to small values of q and as a result of this approximation the mathematical analysis is greatly simplified.

The role of the Ekman boundary layer is central to our treatment of the problem. As far as the motion in the rolls is concerned the Ekman layers only lead to small perturbations which may be disregarded. On the other hand, since the evolution of the geostrophic flow aligned with the applied magnetic field is not influenced either by the Coriolis force or the Lorentz force (except through interactions brought about by the convection rolls themselves) a more subtle force balance is achieved in which Ekman suction plays a prominent part. Indeed it is only through the resulting damping of the geostrophic flow that it is possible in the analysis below to isolate finite-amplitude equilibria.

The outline of the paper is as follows. The mathematical problem is formulated in § 2 and a weakly nonlinear theory of convection is developed for small values of the excess Rayleigh number $R - R_c$ in § 3. A set of equations (see (3.12) below) is derived which relates the amplitudes of the two families of rolls together with the geostrophic flow naturally forced by them. It is shown that for sufficiently small values of $R - R_c$ one family of rolls is stable, while the other is unstable. In § 4 a restricted parameter range (see (4.1) below) is considered. Here equations (3.12) continue to be valid up to and beyond the value of $R - R_c$ at which the single-roll family becomes unstable. After instability sets in a new oscillatory finite-amplitude state emerges whose stability is investigated. In § 5 a more comprehensive study is undertaken of the stability of the single-roll solutions. In particular the region in the λ, τ plane where instability is possible is elucidated (see figure 1). Other than the occurrence of oscillatory geostrophic flow similar to that considered in § 4, steady geostrophic flows can set in when $\lambda < 3^{-\frac{1}{2}}/2$. The stability of the latter flow is investigated in § 6 and the paper ends with a brief discussion of the results in § 7.

2. The governing equations

In this section the mathematical statement of the problem outlined in the introduction is given. Except for the inclusion of the no-slip condition at the boundaries the problem is essentially the same as that considered by RS II. The notation of that paper is followed whenever possible but there are some differences.†

For all time t^*, the system is referred to rectangular Cartesian co-ordinates $Ox^*y^*z^*$, where Oz^* is vertical and Oy^* is aligned with the applied magnetic field. The top and bottom planes are located at $z^* = d$ and 0 respectively. Let the fluid velocity be $\mathbf{u}^*$, the magnetic field be $\mathbf{b}^*$ and the temperature be T. Then upon choosing d/π and $d^2/\kappa\pi^2$ as units of length and time respectively the convection problem may be cast into dimensionless form by the change of variables

$$\mathbf{x}^* = (d/\pi)\,\mathbf{x}, \quad t^* = (d^2/\kappa\pi^2)\,t, \tag{2.1a,b}$$

$$\mathbf{u}^* = (\kappa\pi/d)\,(U\hat{\mathbf{y}}+\mathbf{u}), \quad \mathbf{b}^* = B_0(\hat{\mathbf{y}}+q\mathbf{b}), \tag{2.1c,d}$$

$$T = T_0+\Delta T(-z+\theta)/\pi, \tag{2.1e}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are used to denote the unit vectors in the Ox, Oy, Oz directions respectively. In (2.1c) U, which gives the magnitude of the mean flow in the y direction, is independent of y and z but spatially periodic in x. On the other hand, $\mathbf{u}$, $\mathbf{b}$ and θ are spatially periodic in both x and y (see (3.4) and (3.6) below).

In terms of the dimensionless variables, the governing equations are given by RS II (equation (2.2)). Since the subsequent analysis is restricted to the parameter range

$$E \ll 1, \quad q \ll 1, \tag{2.2a,b}$$

$$\lambda = O(1), \quad R = O(1), \tag{2.2c,d}$$

a number of terms in these equations may be neglected and it is sufficient to consider the simplified system

$$\nabla\,.\,\mathbf{u} = 0, \quad \nabla\,.\,\mathbf{b} = 0, \tag{2.3a,b}$$

$$\nabla^2\theta+\mathbf{u}\,.\,\hat{\mathbf{z}} = \partial\theta/\partial t+\mathbf{u}\,.\,\nabla\theta+U\partial\theta/\partial y, \tag{2.3c}$$

$$\nabla^2\mathbf{b}+\partial\mathbf{u}/\partial y = 0, \tag{2.3d}$$

$$\partial\mathbf{b}/\partial y-\lambda\hat{\mathbf{z}}\times\mathbf{u}+\lambda R\theta\hat{\mathbf{z}}-\nabla p = 0 \tag{2.3e}$$

for the fluctuating quantities $\mathbf{u}$, $\mathbf{b}$, θ and total pressure p. The scaling adopted in (2.1) and the approximations made in deriving (2.3), which are listed below, anticipate the magnitude of the disturbances to be considered in the following sections. Firstly, since q is small, (2.3d, e) are linearized on the basis that the perturbation magnetic field $q\mathbf{b}$ is small. Secondly, since there is considerable Ohmic diffusion (again $q \ll 1$), the advection of magnetic field is neglected in the magnetic-induction equation (2.3d). Thirdly, the inertia and viscous terms in the equation of motion (2.3e), which are clearly negligible in comparison with the Coriolis force ($E \ll 1$), have also been dropped. By contrast no approximations are made at this stage in the heat-conduction equation (2.3c). It is therefore significant that, except for terms describing convection of heat, the system of equations (2.3) is linear.

The boundary conditions that must be satisfied by the solutions of (2.3) are as follows,

$$\mathbf{u}\,.\,\hat{\mathbf{z}} = \theta = \mathbf{b}\,.\,\hat{\mathbf{z}} = \hat{\mathbf{z}}\times\partial\mathbf{b}/\partial z = 0 \quad \text{at} \quad z = 0, \pi. \tag{2.4}$$

† One such example is the choice of direction of the z axis. In RS II it is downward, whereas here it is upward!

Here the no-slip condition $\hat{\mathbf{z}}\times\mathbf{u} = 0$ at $z = 0, \pi$, which leads to the formation of Ekman boundary layers, has been omitted. The reason for this is that we are not concerned with the order $E^{\frac{1}{2}}$ error associated with the neglect of these layers. On the other hand, as indicated in the introduction Ekman suction seriously influences the mean shear, U. Hence when the y component of the equation of motion is averaged and due account of the Ekman jump conditions is taken (e.g. see Moore 1978) the equation

$$\epsilon\, \partial U/\partial t + \Delta U = \lambda^{-1}\, \partial M/\partial x \tag{2.5a}$$

is obtained, where

$$M = \hat{\mathbf{x}}.\langle \mathbf{bb}\rangle.\hat{\mathbf{y}} \tag{2.5b}$$

is a component of the Maxwell stress tensor averaged with respect to both y and z,

$$\epsilon = \tau\Delta, \quad \Delta = E^{\frac{1}{2}}/q, \tag{2.5c}$$

and τ was introduced earlier in (1.6). In (2.5a) the large Coriolis and Lorentz forces present in (2.3e) are absent and a more subtle balance between weaker forces is achieved. In particular the flow is driven by the mean Lorentz force and damped by Ekman suction. The only effects neglected in (2.5a) are the viscous and mean Reynolds stresses in the mainstream flow.

For geophysically relevant values of the parameters, the numbers Δ, τ and ϵ are all likely to be small and ordered such that

$$1 \gg \Delta \gg \tau \gg \epsilon, \tag{2.6}$$

where, of course, only two out of the three are independent. This means that the term ΔU in (2.5a), which was absent in RS II, is important here and plays a central role in the finite-amplitude analysis. Nevertheless, so that comparisons can be made with the earlier work corresponding to ϵ finite and $\Delta = 0$, the subsequent analysis does not stick rigidly to the ordering (2.6) (e.g. see (4.1) below). Finally it must be emphasized that the present calculations also differ from RS II in as much as approximations have been made which depend on the small size of q. The approximation $q \ll 1$, which was not made in RS II, leads to considerable mathematical simplification.

3. Weakly nonlinear convection

The theory of weakly nonlinear convection developed here closely parallels the theory of RS I and RS II. Attention is restricted to the case

$$\lambda < 2/3^{\frac{1}{2}}, \tag{3.1}$$

for which Eltayeb (1972) and RS I have shown instability is characterized by two families of rolls, each with axes oblique to the magnetic field. The two sets of marginal rolls have planform

$$E_{l,m} = \exp i(l\alpha x + m\beta y) \quad (\alpha > 0, \quad \beta > 0), \tag{3.2a}$$

where $l = \pm 1$, $m = \pm 1$ and

$$\alpha^2 = 2 - \beta^2, \quad \beta^2 = 3^{\frac{1}{2}}\lambda. \tag{3.2b, c}$$

The corresponding value of the critical Rayleigh number is

$$R_c = 3^{\frac{3}{2}}. \tag{3.3}$$

We assume that R is close to R_c and so the ensuing finite-amplitude motion is dominated by the roll solutions (3.2), for which a more complete description is given by (3.9) below. Through nonlinear interactions higher harmonics are generated. Analysis of the coupling of the rolls with these harmonics leads to a set of nonlinear equations ((3.12) below), which govern the slow evolution of the amplitude of each roll.

The solution of (2.3) is determined succinctly when **u** and **b** are expressed in terms of their toroidal and poloidal parts

$$\mathbf{u} = \nabla \times \psi \hat{\mathbf{z}} + \nabla \times (\nabla \times \phi \hat{\mathbf{z}}), \quad \mathbf{b} = \nabla \times f \hat{\mathbf{z}} + \nabla \times (\nabla \times g \hat{\mathbf{z}}). \tag{3.4a, b}$$

For then (2.3*a*, *b*) are automatically satisfied and (2.3*c*, *d*, *e*) reduce after integration to

$$\nabla^2 \theta - \nabla_h^2 \phi = \partial\theta/\partial t + N, \tag{3.5a}$$

$$\partial\psi/\partial y + \nabla^2 f = 0, \quad \partial\phi/\partial y + \nabla^2 g = 0, \tag{3.5b, c}$$

$$\partial f/\partial y + \lambda \partial\phi/\partial z = 0, \quad -\partial(\nabla^2 g)/\partial y + \lambda\, \partial\psi/\partial z + \lambda R(\theta - \bar{\theta}) = 0, \tag{3.5d, e}$$

where

$$\nabla^2 = \nabla_h^2 + \partial^2/\partial z^2, \quad N = U \partial\theta/\partial y + \mathbf{u}\,.\nabla\theta. \tag{3.5f, g}$$

The horizontal averages of ψ, ϕ, f and g vanish but the horizontal average of θ, namely $\bar{\theta}(z, t)$, is non-zero for finite amplitude convection. The character of the boundary conditions (2.4) permit solutions of (3.5) and (2.5), which take the form

$$[\psi, f] = \sum_{l,m,n} [\psi, f]_{l,m,n} E_{l,m} \cos nz, \tag{3.6a†}$$

$$[\phi, g, \theta, N] = \sum_{l,m,n} [\phi, g, \theta, N]_{l,m,n} E_{l,m} \sin nz, \tag{3.6b}$$

$$[U, M] = \sum_{l} [U, M]_l E_{l,0}, \tag{3.6c}$$

where $n > 0$, the range of summation runs over integral‡ values of l, m, n and all complex quantities have the property

$$\theta_{-l,-m,n} = \theta^*_{l,m,n}, \tag{3.6d}$$

the star denoting complex conjugate. Thus the governing equations (3.5) reduce to the ordinary differential equation

$$\mathbf{D}_{l,m,n}(0)\, [\phi, \psi, \theta]^T_{l,m,n} = [\Delta\lambda R^{(2)} \theta, 0, -\dot{\theta} - N]^T_{l,m,n}, \tag{3.7a}$$

where the superscript T denotes transpose, the dot denotes differentiation with respect to t ($\dot{\theta} = d\theta/dt$) and

$$\begin{bmatrix} g \\ f \end{bmatrix}_{l,m,n} = \frac{im\beta}{r^2+n^2} \begin{bmatrix} \phi \\ \psi \end{bmatrix}_{l,m,n}, \quad \mathbf{D}_{l,m,n}(p) = \begin{bmatrix} m^2\beta^2 & n\lambda & -\lambda R^{(0)} \\ -(r^2+n^2)\, n\lambda & m^2\beta^2 & 0 \\ -r^2 & 0 & p + (r^2+n^2) \end{bmatrix}, \tag{3.7b, c}$$

$$r^2 = l^2\alpha^2 + m^2\beta^2, \quad R = R^{(0)} + \Delta R^{(2)}, \tag{3.7d, e}$$

provided l and m are not both zero. In addition (2.5) becomes

$$0 = -\epsilon \dot{U}_l - \Delta U_l + il(\alpha/\lambda) M_l, \tag{3.7f}$$

while all the coefficients $N_{l,m,n}$ and M_l are given by (A 1) and (A 2) in appendix A.

† The notation used is that every quantity inside the square brackets is understood to have the subscripts listed following the brackets.

‡ There is an exception. Non-integral values of l in the summation (3.6) are considered in §§ 5 and 6 below.

The solution of

$$\det\{\mathbf{D}_{l,m,n}(p)\} = 0 \tag{3.8}$$

has special significance, since it determines the growth rate p of the free mode (3.2a) according to linear theory. Indeed, when $\lambda < 2/3^{\frac{1}{2}}$, the smallest value of $R^{(0)}$ and the corresponding values of α and β, for which p is zero, are given by (3.3) and (3.2b,c) respectively. In this case the most general solution is

$$\theta = \Delta^{\frac{1}{2}} \sum_{\substack{l=\pm 1\\ m=\pm 1}} \theta^{(1)}_{l,m,1} E_{l,m} \sin z \quad (R^{(0)} = R_c) \tag{3.9a}$$

with similar expressions for the other variables, where

$$\begin{bmatrix} \phi^{(1)} \\ \psi^{(1)} \\ \theta^{(1)} \end{bmatrix}_{1,\pm 1,1} = \begin{bmatrix} 1 \\ 3^{\frac{1}{2}} \\ 2/3 \end{bmatrix} a_{\pm 1}, \quad \begin{bmatrix} g^{(1)} \\ f^{(1)} \end{bmatrix}_{1,\pm 1,1} = \pm i\beta \begin{bmatrix} 1/3 \\ 1/3^{\frac{1}{2}} \end{bmatrix} a_{\pm 1}. \tag{3.9b,c}$$

It represents two distinct rolls whose amplitudes and phases are determined completely by the complex constants $a_{\pm 1}$.

In the case of finite-amplitude convection, Δ is adopted as the expansion parameter. It is supposed that θ, ψ, ϕ, f and g are all of order $\Delta^{\frac{1}{2}}$ and have power-series expansions of the form

$$\theta = \Delta^{\frac{1}{2}}\theta^{(1)} + \Delta\theta^{(2)} + \ldots, \tag{3.10a}$$

where the first-order terms are given by (3.9). Since the Maxwell stress component M is proportional to the square of the magnetic-field strength, it is of order Δ and has an expansion

$$M = \Delta M^{(2)} + \Delta^{\frac{3}{2}} M^{(3)} + \ldots, \tag{3.10b}$$

where $M^{(2)}$ is given by (A 4). Now it transpires that, in the parameter ranges of interest, this stress drives a mean shear U, which is of order $\Delta^{\frac{1}{2}}$ and can be written

$$U = \Delta^{\frac{1}{2}} U^{(1)} + \Delta U^{(2)} + \ldots. \tag{3.10c}$$

At lowest order the direct coupling of the positive (a_+) and negative (a_-) rolls leads to the shear flow

$$U^{(1)} = U^{(1)}_2 E_{2,0} + \text{c.c.} \quad (U^{(1)}_2 = iv/\beta), \tag{3.11}$$

where c.c. denotes the complex conjugate of the expression preceding it. As a result of the assumptions (3.10a,c), the heat convection term (3.5g) is of order Δ and possesses an expansion similar to (3.10b). The first non-zero coefficient $N^{(2)}$ is given by (A 3).

If in addition to the assumptions (3.10), θ and U evolve on a time scale that is long compared with unity, the terms on the right-hand side of (3.7) only lead to small perturbations of the first-order solution (3.9) and so allow (3.7) to be solved iteratively. At second order the evaluation of $M^{(2)}$ and $N^{(2)}$ described in the previous paragraph and outlined in appendix A leads to the conservative system of equations (A 9) which couple the roll amplitudes $a_{\pm 1}$ with the scaled shear flow v. Upon proceeding to third order the analysis of appendix A yields

$$\dot{a}_{\pm 1} = \pm \Delta^{\frac{1}{2}} v a^*_{\mp 1} + \Delta \Phi_{\pm} a_{\pm 1}, \tag{3.12a,b}$$

$$\epsilon \dot{v} = \Delta^{\frac{1}{2}} \mu a_1 a_{-1} + \Delta \Theta v, \tag{3.12c}$$

where
$$\Phi_{\pm} = d - \delta V - 2A_{\pm 1} - (2 + C_S \pm C_A)\, A_{\mp 1}, \tag{3.13a, b}$$
$$\Theta = -1 + k_+ A_1 - k_- A_{-1}, \tag{3.13c}$$
$$A_{\pm 1} = |a_{\pm 1}|^2, \quad V = |v|^2, \tag{3.13d, e}$$
$$d = R^{(2)}/3^{\frac{1}{2}}, \quad \delta = (3 + 4\alpha^2)/32\alpha^4, \quad \mu = 4\alpha\beta/3, \tag{3.13f, g, h}$$
$$k_{\pm} = (\beta/8\alpha^3)\,[(1 - 2\alpha^2)(3 + 4\alpha^2) \mp 2\sqrt{3^{\frac{1}{2}}\alpha\beta(1 + 4\alpha^2)}]/(3 + 8\alpha^2) \tag{3.13i}$$
and the coefficients C_S and C_A are defined by (A 14*a*, *b*). In order to illustrate a few of the key properties of (3.12), which will be analysed in more detail in the following sections, the limit $d \downarrow 0$ is now considered briefly.

The limit $d \downarrow 0$

The derivation of (3.12) was based on the use of Δ as an expansion parameter. There are, however, two other parameters, namely d and τ, which are independent of the magnetic-field strength, that could have been utilized. In fact d is the natural choice and is the one generally adopted in small-amplitude expansions. It has the advantage of isolating the behaviour of the solutions of the governing equations immediately following the bifurcation ($d > 0$) which occurs when $R = R_c$. For our problem it is also the simplest case because the equations may be approximated without appeal to the size of Δ and τ. Instead it is supposed that $A_{\pm 1}$ and v are all of order d. The analysis recovers (3.12), in which correct to order d (3.13*a*, *b*, *c*) are approximated by
$$\Phi_{\pm} = d - 2A_{\pm 1} - (2 + C_S \pm C_A)\, A_{\mp 1} \quad (d > 0), \tag{3.14a, b}$$
$$\Theta = -1. \tag{3.14c}$$

If the free-decay mode $a_{\pm 1} = 0$, $v \propto e^{-t/\tau}$, which corresponds to the simple damping of the shear flow by Ekman suction on the time scale τ, is ignored, attention may be restricted to events which occur on the long time scale d^{-1}. Accordingly $\epsilon\dot{v}$ may be neglected in (3.12*c*) to yield
$$v = \Delta^{-\frac{1}{2}}\mu a_1 a_{-1}. \tag{3.15a}$$
Substitution of (3.15*a*) into (3.12) gives
$$\dot{a}_{\pm 1} = (\pm \mu A_{\mp 1} + \Delta \Phi_{\pm})\, a_{\pm 1}. \tag{3.15b, c}$$
There are three possible steady-state equilibria. The first two are the single-roll solutions
$$A_1^{(0)} = \tfrac{1}{2}d, \quad A_{-1} = V = 0, \tag{3.16a}$$
$$A_{-1}^{(0)} = \tfrac{1}{2}d, \quad A_1 = V = 0. \tag{3.16b}$$
In addition there is a third double-roll solution, for which $A_{\pm 1}$ and V are all non-zero, but this only exists when
$$C_S > |C_A - \mu/\Delta|. \tag{3.17}$$
Since our subsequent analysis is restricted to the case of small Δ, for which the inequality (3.17) fails, this case will be considered no further.

To test the stability of the single-roll solutions (3.16) small perturbations to the basic state are considered and so
$$a_{\pm 1} = a_{\pm 1}^{(0)} + \hat{a}_{\pm 1} \quad \text{and} \quad a_{\mp 1} = \hat{a}_{\mp 1} \tag{3.18}$$
are substituted into (3.15*b*). Linearization yields
$$\dot{\hat{a}}_{\pm 1} = -2\Delta(A_{\pm 1}^{(0)}\, \hat{a}_{\pm 1} + a_{\pm 1}^{(0)2}\, \hat{a}_{\pm 1}^{*}), \tag{3.19a}$$
$$\dot{\hat{a}}_{\mp 1} = -\tfrac{1}{2}\Delta d\{C_S \mp (C_A - \mu/\Delta)\}\, \hat{a}_{\mp 1}. \tag{3.19b}$$

From (3.19a) it follows that perturbations of the primary roll always decay. On the other hand, for all sufficiently small Δ, it is clear from (3.19b) that the positive-roll solution (3.16a) is stable to small perturbations of the secondary negative roll. By contrast, the negative-roll solution (3.16b) is unstable to small perturbations of the secondary positive roll. This result is in partial agreement with the findings of RS II. They found that the negative roll is unstable but they also found that the positive roll is unstable in certain parameter ranges. The discrepancy results from the severe truncation of (3.12), which is applicable in the limit $d \downarrow 0$, keeping Δ and τ finite, and will be resolved in the more detailed analysis of the following sections.

4. Oscillatory shear flow

In view of the results of the preliminary investigation of the limit $d \downarrow 0$ at the end of § 3, attention is focused here and in § 5, below, upon the stability of the positive roll (3.16a) when d is of order unity. In § 5 a general linear-stability calculation is undertaken to clarify the parameter ranges, in which the positive roll is unstable. It indicates that, when τ is large and λ lies in the interval (1.5), the most readily excited overstable modes are correctly described by (3.12). Since the applicability of (3.12) is not restricted to the linear regime, they are utilized in this section to investigate the finite-amplitude oscillatory shear flow, which is set up at the post-bifurcation stage. In order to obtain simple solutions of (3.12) it is expedient to make the additional assumption that ϵ is small. Consequently throughout this section it is supposed that

$$d = O(1), \quad 1 \gg \epsilon \gg \Delta. \tag{4.1a,b}$$

Since ϵ equals $\tau\Delta$, the final inequality is simply a statement that τ is large.

In the parameter range (4.1), $a_{\pm 1}$ and v satisfy the simplified system (A 9) at lowest order. A number of integrations can be made which yield three constants of the motion. They are

$$S_{\pm 1} = \mu A_{\pm 1} \mp \epsilon V, \tag{4.2a}$$

$$h = -\tfrac{1}{2} i \tau^{\frac{1}{2}} (v^* \dot{v} - v \dot{v}^*). \tag{4.2b}$$

Elimination of $a_{\pm 1}$ from (A 9) then yields the single second-order equation

$$\tau \ddot{v} + (S + 2\epsilon V) v = 0, \tag{4.3a}$$

where
$$S = S_1 - S_{-1}. \tag{4.3b}$$

Integration of (4.3a) leads to

$$\tfrac{1}{4}\tau \dot{V}^2 = -h^2 + 2EV - SV^2 - \epsilon V^3, \tag{4.3c}$$

where (A 9) may be used to show that the constant of integration E is given by

$$\epsilon E = \tfrac{1}{2} S_1 S_{-1}. \tag{4.3d}$$

According to (4.3a) v can execute nonlinear finite-amplitude oscillations, which define orbits in the complex v-plane. In addition to S, (4.3c) indicates that the principal characteristics of the orbits are defined by the constants h and E which are analogous to angular momentum and energy in classical orbit theory.

Exact solutions of (4.3c) exist in terms of elliptic functions. Nevertheless, when S is positive and provided $\epsilon V/S$ remains small, the solution of (4.3a) only deviates slightly from simple harmonic motion with frequency $(S/\tau)^{\frac{1}{2}}$. In this case it is more convenient to represent the solution by the rapidly convergent Fourier series

$$V = V_0 + \{V_2 e^{2i\omega t} + \epsilon V_4 e^{4i\omega t} + \text{c.c.}\} + O(\epsilon^2), \tag{4.4}$$

where $2\pi/\omega$ is the period of oscillation of the complex velocity v. Substitution of (4.4) into (4.3c) and equating coefficients yield

$$\tau\omega^2 = S + 3\epsilon E/S + O(\epsilon^2), \tag{4.5a}$$

$$V_0 = E/S - (3\epsilon/4S^3)(3E^2 - h^2 S) + O(\epsilon^2), \tag{4.5b}$$

$$|V_2|^2 = (E^2 - h^2 S)/4S^2 - (\epsilon E/4S^4)(4E^2 - 3h^2 S) + O(\epsilon^2), \tag{4.5c}$$

$$V_4 = V_2^2/2S + O(\epsilon). \tag{4.5d}$$

It is perhaps worth noting at this point that, when $E = v = 0$, the cases $S > 0$ and $S < 0$ correspond to the positive- and negative-roll solutions (3.16a, b) respectively. According to (4.3a) the positive roll is neutrally stable, while the negative roll is unstable. Nevertheless, even when $S < 0$, v does not grow indefinitely because of the term $2\epsilon V$ in (4.3a). Instead after a finite perturbation of h, E and v it executes large-amplitude oscillations, which cannot be adequately described by (4.4) and (4.5).

When the terms of order Δ in (3.12) are taken into account (4.3a, c) must be modified and the term $\dot{V}^2$ in (4.3c) is replaced by $(\dot{V} - 2\tau^{-1}\Theta V)^2$. In addition $S_{\pm 1}$ and h are no longer constant but satisfy

$$\dot{S}_{\pm 1} = 2\Delta\{\Phi_\pm S_{\pm 1} \pm (\epsilon\Phi_\pm - \Theta) V\}, \tag{4.6a, b}$$

$$\dot{h} = \Delta\{\Phi_+ + \Phi_- + \epsilon^{-1}\Theta\} h, \tag{4.6c}$$

where, from (3.13) and (4.2a),

$$\Phi_\pm = d - \mu^{-1}\{2S_{\pm 1} + (2 + C_S \pm C_A) S_{\mp 1}\} - \{\delta \mp (\epsilon/\mu)(C_S \pm C_A)\} V, \tag{4.7a, b}$$

$$\Theta = -1 + \mu^{-1}\{k_+ S_1 - k_{-1} S_{-1}\} + (\epsilon/\mu)(k_+ + k_-) V. \tag{4.7c}$$

Since V fluctuates rapidly, $S_{\pm 1}$ and h do also. Their fluctuating parts, however, are small and are dominated by their mean parts which evolve slowly. For this reason (4.6) is averaged over the period of oscillation $2\pi/\omega$ and the approximation is made that

$$\overline{S}_{\pm 1} = S_{\pm 1}, \quad \overline{h} = h, \tag{4.8}$$

where the average is denoted by the bar. As a result, the only fluctuating quantities in (4.6), which must be averaged, are V and V^2. Their values are

$$\overline{V} = V_0, \quad \overline{V^2} = V_0^2 + 2|V_2|^2 + O(\epsilon^2), \tag{4.9a, b}$$

where V_0 and $|V_2|^2$ are defined by (4.5).

The secular behaviour of small-amplitude oscillations in the neighbourhood of the positive-roll solution may be investigated by supposing that

$$S_1 = S_1^{(0)} + \epsilon S_1^{(1)} + O(\epsilon^2), \quad S_{-1} = \epsilon S_{-1}^{(1)} + \epsilon^2 S_{-1}^{(2)} + O(\epsilon^3). \tag{4.10}$$

The corresponding values of E and S are determined by (4.3b, d) and so (4.5) and (4.9) yield

$$\overline{V} = \tfrac{1}{2} S_{-1}^{(1)} + O(\epsilon), \tag{4.11a}$$

$$\overline{V^2} = (3S_1^{(0)} S_{-1}^{(1)2} - 4h^2)/8S_1^{(0)} + O(\epsilon). \tag{4.11b}$$

To leading order in powers of ϵ the average of (4.6) simplifies considerably and reduces to

$$\dot{S}_1^{(0)} = \Delta\left\{\left(2d - \frac{4}{\mu}S_1^{(0)} - \delta S_{-1}^{(1)}\right)S_1^{(0)} + \left(1 - \frac{k_+}{\mu}S_1^{(0)}\right)S_{-1}^{(1)}\right\}, \quad (4.12a)$$

$$\dot{S}_{-1}^{(1)} = -(\Delta/\epsilon)\left(1 - \frac{k_+}{\mu}S_1^{(0)}\right)S_{-1}^{(1)}, \quad (4.12b)$$

$$\dot{h} = -(\Delta/\epsilon)\left(1 - \frac{k_+}{\mu}S_1^{(0)}\right)h. \quad (4.12c)$$

The positive-roll solution (3.16a) corresponds to the stationary solution

$$S_1^{(0)} = \tfrac{1}{2}\mu d, \quad S_{-1}^{(1)} = h = 0. \quad (4.13)$$

According to (4.12) it is unstable when

$$0 < 2/k_+ < d \quad (4.14)$$

and stable otherwise. Evidently instability is only possible for some value of $d(>0)$, when $k_+ > 0$. This condition on k_+ is exactly the same as the inequality (4.23) of RS II, which holds whenever λ lies in the interval (1.5).

When the inequality (4.14) is satisfied a new finite-amplitude equilibrium is possible with

$$S_1^{(0)} = \mu/k_+, \quad S_{-1}^{(1)} = (2/\delta)(d - 2/k_+), \quad h = \text{constant}. \quad (4.15)$$

It is composed of a primary and secondary flow. The former consists of the single positive roll, whose amplitude is fixed independent of the Rayleigh number such that $A_1 = 1/k_+$. The latter consists of the shear flow, which is comparable in magnitude to the primary roll, together with additional order ϵ contributions of both positive and negative rolls. According to (4.5a) and (4.15), this secondary flow oscillates with frequency

$$\omega = (\mu/k_+\tau)^{\frac{1}{2}} + O(\epsilon\tau^{-\frac{1}{2}}). \quad (4.16)$$

When small perturbations are superimposed on this basic state so that

$$S_1^{(0)} = \mu/k_+ + \epsilon^{\frac{1}{2}}s_1, \quad S_{-1}^{(1)} = (2/\delta)(d - 2/k_+) + s_{-1}, \quad (4.17)$$

linearization of (4.12a, b) yields

$$\tau^{\frac{1}{2}}\dot{s}_1 = -\Delta^{\frac{1}{2}}\{(\mu\delta/k_+)s_{-1} + 2\epsilon^{\frac{1}{2}}[2/k_+ + (k_+/\mu\delta)(d - 2/k_+)]s_1\}, \quad (4.18a)$$

$$\tau^{\frac{1}{2}}\dot{s}_{-1} = 2\Delta^{\frac{1}{2}}(k_+/\mu\delta)(d - 2/k_+)s_1. \quad (4.18b)$$

At lowest order the solutions describe oscillations with frequency

$$\Delta^{\frac{1}{2}}\{(2d - 4/k_+)/\tau\}^{\frac{1}{2}}. \quad (4.18c)$$

When the $\epsilon^{\frac{1}{2}}$ term in (4.18a) is not neglected, however, it becomes apparent that these oscillations are lightly damped and decay on the longer time scale Δ^{-1}.

The above calculations establish the stability of the new equilibrium (4.15) but give no indication of the value taken by h, because the coefficient of h in (4.12c) is zero. A higher-order theory is therefore required which considers the equations for $S_1^{(1)}$ and $S_{-1}^{(2)}$ in the expansions (4.10), where the first-order terms $S_1^{(0)}$ and $S_{-1}^{(1)}$ are given by (4.15). Since h and consequently $S_1^{(1)}$ and $S_{-1}^{(2)}$ evolve on the long time scale Δ^{-1}, the term $\dot{S}_{-1}^{(2)}$ in (4.6b) is negligible and so the average of that equation yields

$$\epsilon^{-1}\overline{\Phi_- S_{-1}} - \overline{\Phi_- V} + \epsilon^{-1}\overline{\Theta V} = O(\epsilon). \quad (4.19)$$

The basis for this approximation is suggested by the comparison of the right-hand sides of (4.12*a*, *b*), where the latter is larger by a factor ϵ^{-1}. Formally at any rate (4.19) determines $S_1^{(1)}$ in terms of h, while (4.6*a*) determines $S_{-1}^{(2)}$ in terms of $S_1^{(1)}$, $\dot{S}_1^{(1)}$ and h. With the aid of (4.19) the analysis of appendix B shows that (4.6*c*) reduces to

$$\dot{h} = -\tfrac{1}{2}\Delta K\left(\frac{E^2 - h^2 S}{ES}\right)h, \tag{4.20a}$$

where to the order of accuracy considered E and S are constants and

$$K = \delta + \frac{k_+ + k_-}{\mu} = \frac{(3+4\alpha^2)(9-4\alpha^2)}{32\alpha^4(3+8\alpha^2)} \tag{4.20b}$$

is positive for all α^2 in the interval $0 \leqslant \alpha^2 \leqslant 2$ of interest.

According to (4.20*a*) there are two stationary equilibria. One is characterized by

$$h^2 = E^2/S \tag{4.21a}$$

for which

$$V_0 = E/S, \quad V_2 = 0. \tag{4.21b, c}$$

The corresponding complex velocity is

$$\beta U^{(2)} = iv = V_0^{\frac{1}{2}} e^{\pm i\omega t} \tag{4.22a}$$

and so the resulting shear flow velocity (3.11) describes one of the two travelling waves

$$U = (2/\beta)\, V_0^{\frac{1}{2}} \cos(2\alpha x \pm \omega t). \tag{4.22b}$$

The other equilibrium is characterized by

$$h = 0, \tag{4.23a}$$

for which

$$2|V_2| = V_0 = E/S. \tag{4.23b}$$

The corresponding complex velocity is

$$\beta U^{(2)} = iv = (2V_0)^{\frac{1}{2}} \cos\omega t \tag{4.24a}$$

and so in this case the shear flow velocity (3.11) is described by the standing wave

$$U = (2/\beta)(2V_0)^{\frac{1}{2}} \cos\omega t \cos 2\alpha x. \tag{4.24b}$$

Since K is positive, (4.20*a*) indicates that the travelling-wave solution (4.22) is unstable, while the standing-wave solution (4.24) is stable.

In view of the result (4.14) it is apparent that the analysis of this section relates closely to the problem considered in RS II. There Ekman suction was ignored and this corresponds to the omission of the term -1 in the expression (3.13*c*) for Θ. Inspection of the governing equations (3.7) indicates that, when Ekman suction is included, this term is only negligible in the limit $\Delta \to 0$ with ϵ and Δd kept finite. Since all the equilibria isolated in this section rely on this damping of the geostrophic shear for either their existence or stability, it is clear that the case $d = O(\Delta^{-1})$ lies outside the scope of the present calculations. Despite the fact that our analysis can say nothing about this inviscid limit, it does isolate some of the key physical processes which are of fundamental importance to the problem.

To summarize, the positive-roll solution (3.16a) is stable when $k_+ < 0$ for all order one values of d but when $k_+ > 0$ it is stable only when $d < 2/k_+$. In these parameter ranges all shear flow perturbations are damped and the squared amplitude A_1 of the positive roll is simply proportional to the excess Rayleigh number. Once d exceeds the value $2/k_+ (> 0)$ this solution is unstable because the corresponding value of Θ is negative. This means that the term $\Delta\Theta v$ in (3.12c) rather than damping shear flow perturbations causes them to grow. To achieve a new stable equilibrium the amplitude of the positive roll remains roughly constant in such a way that its tendency to accelerate the shear flow exactly balances damping by Ekman suction (see (3.12c) and (3.13c): $A_1 \doteqdot 1/k_+$). On the other hand, as the Rayleigh number is increased the amplitude of the positive roll can only remain fixed if Φ_+ is kept small (see (3.12a) and (3.13a): $\bar{V} = (d - 2/k_+)/\delta$). The oscillatory character of the secondary flow, which occurs after the bifurcation at $d = 2/k_+$, is imposed by the relatively large terms of order $\Delta^{\frac{1}{2}}$ in (3.12) which provide a most effective coupling between the shear flow and the two rolls. Even for order one values of d the oscillatory shear flow is comparable in magnitude to the primary convection roll. Consequently when d is large the shear flow must predominate and the positive roll is then relatively insignificant. Of course, the analysis is not valid when d is of order Δ^{-1} but we may speculate that long before this value is achieved the basic flow proposed in this section will be subject to yet further secondary instabilities.

5. The stability of the finite-amplitude roll

The stability analysis of the positive roll in § 4 is not generally valid for two reasons. Firstly, in addition to the assumption $\Delta \ll 1$, which will also be made here, the analysis required that $\tau \gg 1$. Secondly, only a special class of disturbances to the positive roll was investigated and no attempt was made to investigate stability with respect to arbitrary perturbations. To rectify these deficiencies the analysis of this section is focused on the parameter range

$$\Delta \ll 1, \quad \tau = O(1). \tag{5.1}$$

Here, guided by the results of § 4, it is reasonable to anticipate that the most unstable disturbances are not of the convective type but rather take the form of geostrophic flow perturbations. One further point which emerges from the analysis is that the mechanism for the instability is associated with the term $\Delta k_+ A_1 v$ in (3.12c), which does not depend in any direct way on the interaction of the shear with the negative roll. There is therefore no reason why attention should be focused upon the shear (3.11), whose choice was motivated by the fact that it provided the natural coupling between the positive and negative rolls. Instead throughout this section the stability of the positive roll is investigated with respect to the arbitrary shear flow perturbation

$$U^{(1)} = U^{(1)}_{2L} E_{2L,0} + \text{c.c.} \quad (U^{(1)}_{2L} = iv/\beta), \tag{5.2}$$

where L is some positive number not necessarily equal to unity.

The interaction of the mean shear flow (5.2) with the positive roll leads to convection of heat, which may be quantified with (A 1c) and (3.9) to give

$$N^{(2)} = -\tfrac{2}{3}(va_1 E_{2L+1,1} - v^* a_1 E_{-2L+1,1}) \sin z + \text{c.c.} \tag{5.3}$$

As a result the forced mode, which is of order Δ, can be expressed in the form

$$\theta^{(2)} = \tfrac{2}{3}(\vartheta_+ E_{2L+1,1} + \vartheta_- E_{-2L+1,1}) \sin z + \text{c.c.} \tag{5.4}$$

It interacts with the original positive roll to produce a mean Maxwell stress

$$M^{(3)} = M^{(3)}_{2L} E_{2L,0} + \text{c.c.} \tag{5.5}$$

and the associated force drives the original shear flow perturbation (5.2). As a result of the various interactions the equations governing the perturbations $\vartheta_\pm$ and v are

$$\dot{\vartheta}_+ + \mu_+ \vartheta_+ = v a_1 + O(\Delta), \tag{5.6a}$$

$$\dot{\vartheta}_- + \mu_- \vartheta_- = -v^* a_1 + O(\Delta), \tag{5.6b}$$

$$\tau \dot{v} + v = \mu(\lambda_+ \vartheta_+ a_1^* + \lambda_- \vartheta_-^* a_1) + O(\Delta), \tag{5.6c}$$

where

$$\mu_\pm = \tilde{\mu}(\pm L), \quad \tilde{\mu}(L) = \frac{8L^2(1+L)^2 \alpha^4}{3+2L(1+L)\alpha^2} \geqslant 0, \tag{5.7a}$$

$\lambda_\pm = \pm \tilde{\lambda}(\pm L)$,

$$\tilde{\lambda}(L) = 3L \frac{(1-(1+L)\alpha^2)(3+2L(1+L)\alpha^2) - 3^{\frac{1}{2}} \alpha\beta(1+L)(1+2L(1+L)\alpha^2)}{(3+4L(1+L)\alpha^2)(3+2L(1+L)\alpha^2)}. \tag{5.7b}$$

The details of the calculations leading to (5.6) are outlined in appendix C. In the stability calculations which follow, it is assumed that the amplitude a_1 of the positive-roll is constant. Nevertheless the finite amplitude interaction of the shear flow v with the temperature perturbations $\vartheta_\pm$ may cause A_1 to evolve over a long time scale and not to stay close to its equilibrium values $\frac{1}{2}d$. In appendix C the equation

$$\dot{a}_1 = \Delta\{(v\vartheta_- - v^* \vartheta_+) + (d - 2A_1) a_1\} + O(\Delta^2) \tag{5.8}$$

governing a_1 is derived and it is used in § 6 below in conjunction with (5.6) to investigate a steady finite-amplitude state that may ensue once the positive roll is no longer stable.

When $L = 1$, it is possible to make a comparison of equations (5.6) and (5.8) with (3.12). In this case the coefficients $\mu_\pm$ and $\lambda_\pm$ are

$$\mu_- = 0, \quad \mu_+ = 1/\delta, \tag{5.9a,b}$$

$$\lambda_- = 1, \quad \lambda_+ = k_+/\mu\delta \tag{5.9c,d}$$

and

$$\Delta^{\frac{1}{2}} \vartheta_- = a_{-1}^*. \tag{5.9e}$$

The only term remaining in (5.6) and (5.8), which does not appear in (3.12), is $\dot{\vartheta}_+$ in (5.6*a*). It was neglected in § 3 on the basis that the time scale is long. With this term ignored (3.12) reduces to (5.6) and (5.8) if

$$\Phi_+ = d - \delta V - 2A_1, \quad \Phi_- = 0, \quad \Theta = -1 + k_+ A_1. \tag{5.10}$$

The other terms appearing in (3.13*a–c*) are not important in this section because the *amplitude of the secondary convection is so small. They are only important in a* higher-order theory.

Upon linearization equations (5.6) and (5.8) decouple. Hence (5.8) becomes independent of $\vartheta_\pm$ and v and the simplified equation clearly demonstrates the stability of the equilibrium $A_1 = \frac{1}{2}d$. With a_1 constant the linear equations (5.6) have solutions proportional to e^{pt}, where p is a root of the cubic

$$\tau p + 1 = A_1 \mu \left(\frac{\lambda_+}{p+\mu_+} - \frac{\lambda_-}{p+\mu_-}\right). \tag{5.11}$$

Evidently when the amplitude a_1 of the convection roll is small the mean Lorentz force is insignificant and (5.11) indicates the perturbation decays at the rate $1/\tau$. As the Rayleigh number and amplitude a_1 increase, the shear flow may become unstable in one of two ways. The simplest case is steady flow, $p = 0$, and then the amplitude of the roll together with the other coefficients appearing in (5.11) must be such that

$$\frac{1}{A_1} = \mu\left(\frac{\lambda_+}{\mu_+} - \frac{\lambda_-}{\mu_-}\right). \tag{5.12}$$

Evidently instability is only possible for some value of A_1 when the right-hand side of (5.12) is positive. The parameter range for which this occurs appears to be limited to small values of both L and λ. Indeed the largest value of λ, for which the positivity requirement can be met for some value of L, corresponds to the limiting case $L = 0$. In this limit, we have

$$\frac{\lambda_+}{\mu_+} - \frac{\lambda_-}{\mu_-} = \frac{(2\alpha^2 - 3)(3(\alpha^2+2) - 3^{\frac{1}{2}}\alpha\beta)}{12\alpha^4}, \tag{5.13}$$

which vanishes when $\alpha^2 = \frac{3}{2}$. It follows that the exchange of stabilities occurs for some value of A_1 only when

$$\lambda < 3^{-\frac{1}{2}}/2. \tag{5.14}$$

If, on the other hand, instability sets in as overstability, the p is purely imaginary,

$$p = i\omega, \tag{5.15a}$$

and the real and imaginary parts of (5.11) yield

$$\frac{1}{A_1} = -\frac{\mu}{\mu_+ + \mu_-}\left(\frac{\lambda_+}{\tau\mu_- + 1} - \frac{\lambda_-}{\tau\mu_+ + 1}\right), \tag{5.15b}$$

$$\omega^2 = -\frac{\lambda_-\mu_+^2(\tau\mu_- + 1) - \lambda_+\mu_-^2(\tau\mu_+ + 1)}{\lambda_-(\tau\mu_- + 1) - \lambda_+(\tau\mu_+ + 1)}. \tag{5.15c}$$

The region in parameter space in which overstability is possible for some value of A_1 is determined by the requirement that the right-hand sides of (5.15*b*, *c*) are both positive. It is illustrated in figure 1, where the stability boundary, which corresponds to $1/A_1 = 0$, was computed numerically. It is perhaps of some interest to note that the value of L for the critical mode on this boundary increases from zero at

$$\lambda \doteqdot 1{\cdot}03098, \quad 1/\tau = 0 \tag{5.16a}$$

to infinity at

$$\lambda = 2/3^{\frac{1}{2}}, \quad 1/\tau = 0. \tag{5.16b}$$

When $\tau \gg 1$ and L is fixed not close to unity, it is clear from (5.15*b*) that instability first sets in when A_1 is of order τ. By contrast, when $|L-1| \ll 1$, (5.15) with the help of (5.9) indicates that

$$A_1 = 1/k_+, \quad \omega = (\mu/k_+\tau)^{\frac{1}{2}}. \tag{5.17}$$

Consequently when λ lies in the interval (1.5), k_+ is positive and the order one value of A_1 given by (5.17) yields the critical value of the amplitude appropriate to the onset of instability in agreement with (4.15) and (4.16) of § 4.

Inspection of figure 1 indicates that overstability can only occur for small values of $1/\tau$. Furthermore, since the parameter ranges in which the exchange of stabilities and overstability are possible are mutually distinct, there is never competition for the

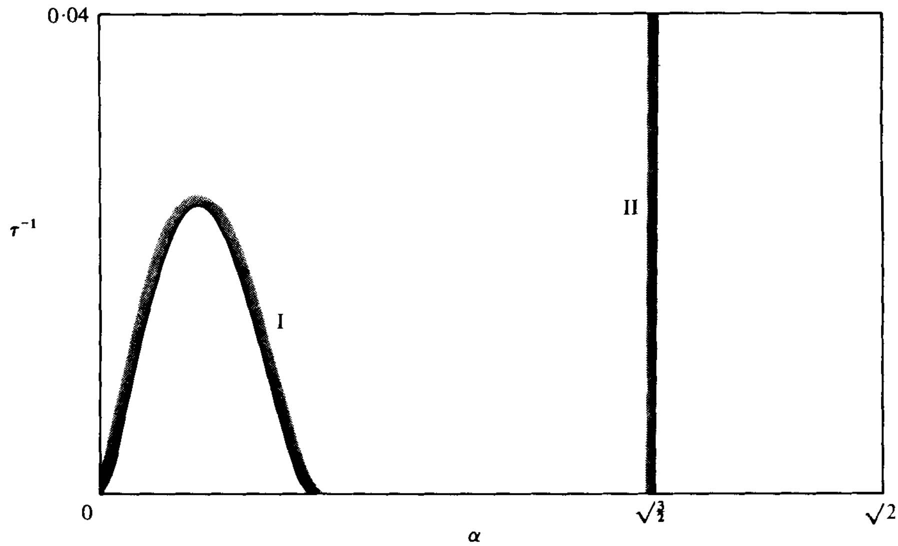

FIGURE 1. The curves I and II define the stability boundaries for overstability and exchange of stability in the α, τ^{-1} plane. Here $\alpha = (2-3^{\frac{1}{2}}\lambda)^{\frac{1}{2}}$ rather than λ is chosen as a co-ordinate to stretch out the region in which overstability is possible.

preferred mode. One other interesting comparison between the two modes of instability is worth noting. It is that the steady modes (5.12) are characterized by

$$\frac{\lambda_-}{\mu_-} < \frac{\lambda_+}{\mu_+} < 0 \qquad (5.18a)$$

whereas the overstable modes (5.15) are characterized by

$$0 < \frac{\lambda_-}{\tau\mu_+ + 1} < \frac{\lambda_+}{\tau\mu_- + 1}. \qquad (5.18b)$$

This means that the ϑ_- perturbations are responsible for driving the steady modes, in contrast with the overstable modes which are driven by the ϑ_+ perturbations. Once instability sets in new finite-amplitude states are possible. Those oscillatory states characterized by $\tau \gg 1$ and $L = 1$ were analysed in § 4. On the other hand, the steady states characterized by small values of λ and L are discussed in the following section.

6. Steady shear flow

When $\lambda < 3^{-\frac{1}{2}}/2$ and when A_1 exceeds its critical value A_{1c} (say) defined by (5.12), a new steady-state equilibrium solution of (5.6) and (5.8) is possible in which $\vartheta_\pm$ and V take the values

$$\vartheta_{+c} = \mu_+^{-1} v_0 a_{1c}, \quad \vartheta_{-c} = -\mu_-^{-1} v_0^* a_{1c}, \qquad (6.1a, b)$$

$$|v_0|^2 = V_0 = (d - 2A_{1c}) \Big/ \left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right). \qquad (6.1c)$$

To test stability we set

$$\vartheta_\pm = \vartheta_{\pm c} + \tilde{\vartheta}_\pm e^{pt}, \quad a_1 = a_{1c} + \tilde{a}_1 e^{pt}, \quad v = v_0 + \tilde{v} e^{pt} \qquad (6.2a, b, c)$$

together with similar expressions for $\vartheta_\pm^*$, a_1^*, v^*, where, for example,

$$a_1^* = a_{1c}^* + \tilde{a}_1^* e^{pt}. \qquad (6.2d)$$

In other words, the complex conjugate of the function is taken *without* taking the complex conjugate of p. The expressions (6.2) are substituted into (5.6) and (5.8) which are then linearized with respect to the constants marked with a tilde. Accordingly (5.6*a*, *b*) yield

$$\tilde{\vartheta}_+ = (v_0\tilde{a}_1 + a_{1c}\tilde{v})/(p+\mu_+), \quad \tilde{\vartheta}_- = -(v_0^*\tilde{a}_1 + a_{1c}\tilde{v}^*)/(p+\mu_-), \tag{6.3}$$

which together with (5.6*c*) and (5.8) lead to a pair of equations relating a_1, a_1^*, v and v^*.

Two types of mode are possible, which can be distinguished by the values of

$$\tilde{V} = v_0^*\tilde{v} + v_0\tilde{v}^*, \quad \tilde{A}_1 = a_{1c}^*\tilde{a}_1 + a_{1c}\tilde{a}_1^*, \tag{6.4a, b}$$

$$\tilde{h} = -\tfrac{1}{2}ip\tau^{\frac{1}{2}}(v_0^*\tilde{v} - v_0\tilde{v}^*), \quad \tilde{\mathscr{A}}_1 = -\tfrac{1}{2}ip\tau^{\frac{1}{2}}(a_{1c}^*\tilde{a}_1 - a_{1c}\tilde{a}_1^*), \tag{6.4c, d}$$

where $\mathscr{A}_1$ is a new variable but h was defined earlier by (4.2*b*). One mode is characterized by $\tilde{V} = \tilde{A}_1 = 0$. It corresponds to the situation in which the amplitude of the basic state remains unchanged but the phase of the complex constants defining it vary with time. In this case (5.6*c*), (5.8), (6.4*c*, *d*) yield

$$\frac{1}{p}\left\{\tau p + 1 - \mu A_{1c}\left(\frac{\lambda_+}{p+\mu_+} - \frac{\lambda_-}{p+\mu_-}\right)\right\}\tilde{h} = \frac{\mu V_0}{p}\left\{\frac{\lambda_+}{p+\mu_+} - \frac{\lambda_+}{\mu_+} + \frac{\lambda_-}{p+\mu_-} - \frac{\lambda_-}{\mu_-}\right\}\tilde{\mathscr{A}}_1, \tag{6.5a}$$

$$\frac{1}{p}\left\{p - \Delta(d - 2A_{1c}) + \Delta V_0\left(\frac{1}{p+\mu_+} + \frac{1}{p+\mu_-}\right)\right\}\tilde{\mathscr{A}}_1 = -\frac{\Delta A_{1c}}{p}\left\{\frac{1}{p+\mu_+} - \frac{1}{\mu_+} - \frac{1}{p+\mu_-} + \frac{1}{\mu_-}\right\}\tilde{h}, \tag{6.5b}$$

which after elimination of $\tilde{h}$ and $\tilde{\mathscr{A}}$ leads to a dispersion relation for p. Upon substitution of the identities (5.12) and (6.1) it reduces further to the quadratic equation

$$\mathscr{L}(p) = -\Delta V_0\left\{\tau\left[\frac{1}{\mu_+(p+\mu_+)} + \frac{1}{\mu_-(p+\mu_-)}\right] - 2\mu A_{1c}\frac{\lambda_+ - \lambda_-}{\mu_+\mu_-(p+\mu_+)(p+\mu_-)}\right\}, \tag{6.6a}$$

where
$$\mathscr{L}(p) = \tau + \mu A_{1c}\left\{\frac{\lambda_+}{\mu_+(p+\mu_+)} - \frac{\lambda_-}{\mu_-(p+\mu_-)}\right\}. \tag{6.6b}$$

At lowest order the two values of p satisfy

$$p\mathscr{L}(p) = 0 \tag{6.7}$$

which is simply (5.11) with A_1 given by (5.12). By definition of the critical state the two non-zero roots of (6.7) must have negative real parts and so stability is guaranteed.

The other class of solutions are characterized by $\tilde{h} = \tilde{\mathscr{A}} = 0$ and correspond to amplitude fluctuations with no phase change. As above a pair of coupled equations similar to (6.5) can be obtained which yield the quartic equation

$$p\{p - 2\Delta(d - 4A_{1c})\}\mathscr{L}(p) = -\Delta V_0(\tau p + 2)\left\{\frac{1}{p+\mu_+} + \frac{1}{\mu_+} + \frac{1}{p+\mu_-} + \frac{1}{\mu_-}\right\} \tag{6.8}$$

for p. Two of the roots are of order unity and are again given to leading order in Δ by $\mathscr{L}(p) = 0$. They correspond to a pair of decaying modes. The remaining two roots are small and can be expressed in the form

$$p = \pm i\Delta^{\frac{1}{2}}\omega^{(1)} + \Delta p^{(2)} + O(\Delta^{\frac{3}{2}}), \tag{6.9a}$$

where
$$\omega^{(1)2} = \frac{4V_0}{\mathscr{L}(0)}\left(\frac{1}{\mu_+}+\frac{1}{\mu_-}\right), \tag{6.9b}$$
$$p^{(2)} = -4A_{1c}(1-V_0/V_c) \tag{6.9c}$$
and V_c is a complicated function independent of V_0. Since the product of the non-zero roots of (6.7) is positive, it follows that $\mathscr{L}(0) > 0$. In addition $\mu_\pm$ are also positive and so the frequency $\omega^{(1)}$ is real. Evidently if $V_c < 0$, the equilibrium is stable for all order one values of V_0 but if $V_c > 0$ the equilibrium becomes unstable once V_0 exceeds the critical value V_c.

Though the steady finite-amplitude state analysed in this section is quite distinct from the oscillatory motion investigated in § 4 there is one important similarity. It is that, once the shear flow develops, the amplitude of the positive roll remains constant as the Rayleigh number is increased. Specifically in the steady case when a_1 takes its critical value, the growth rates for the three normal modes of (5.6) are given by the roots of (6.7). Two of the modes decay, while the third is neutrally stable. With the addition of the amplitude equation (5.8), a_1 is no longer held constant and the results are modified. In fact the preceding analysis has shown that the two decaying modes predicted by (5.6) split into four decaying modes, while the neutral mode executes slow oscillations on a time scale of order $\Delta^{-\frac{1}{2}}$. These oscillations may either grow or decay on a yet longer time scale of order Δ^{-1} depending on the values taken by V_0 and V_c (see (6.9)).

7. Discussion

The instability of the single-roll solutions is at first sight surprising, since it contradicts the following intuitive notion. The idea is that, as a result of single-roll convection, magnetic field perturbations are produced linking planes x = constant, on which the geostrophic flow takes a constant value, one with another. Consequently one anticipates that the geostrophic flow would have an Alfvén wave character similar to the torsional oscillations described in the introduction. Furthermore once dissipative processes are accounted for this motion would eventually decay. The reason for the failure of this simple picture is that we have been considering in this paper very small convective velocities which do not lead to transverse magnetic fields of sufficient size to make this mechanism operative. Instead a more subtle forcing of the geostrophic flow is achieved which is most readily traced through the terms on the right-hand side of (5.11). The processes which lead to the force are as follows. The introduction of the shear proportional to $E_{2L,0}$ (see (5.2)) leads to convection of the heat perturbations originally generated by the convective rolls. The new perturbations force two secondary rolls whose planforms are $E_{2L+1,1}$ and $E_{-2L+1,1}$ and whose structures are determined from (5.4). It is the contribution to the Lorentz force, resulting from the magnetic field perturbations associated with one of the two secondary rolls when they are superimposed upon the initial $E_{1,1}$ field perturbations due to the original rolls that forces the geostrophic motion and so leads to instability. The $E_{-2L+1,1}$ roll is responsible for the exchange of stabilities, while the $E_{2L+1,1}$ roll is responsible for over-stability ($L > 0$).

Production of geostrophic flow by this instability is evidently of some importance, since it provides a basic mechanism capable of limiting the growth of convective motions. In particular, when the Rayleigh number is increased beyond the value appropriate to the geostrophic shear flow instability, the amplitude of the convection remains roughly constant, while the shear velocity continues to increase. Though the range of validity of these results is severely limited by the small parameter expansion scheme adopted, the qualitative picture may well hold true for quite moderate increments of the Rayleigh number. That being so the flow is dominated by geostrophic motions which depend for their existence on relatively slow convective velocities. How these results are modified in more realistic geometries or by the addition of mean meridional magnetic fields (i.e. with components in the x, z plane) is of considerable interest but outside the scope of the present analysis.

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Appendix A

Substitution of (3.6) into (3.4a) and (3.5g) yields

$$N_{l,m,n} = \mathcal{N}^{(+)}_{l,m,n} + \mathcal{N}^{(-)}_{l,m,n} + \mathcal{N}_{l,m,n}, \tag{A 1a}$$

where

$$\mathcal{N}^{(\pm)}_{l,m,n} = \tfrac{1}{2} \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m \\ |n_1 \pm n_2|=n}} \operatorname{sgn}(n_1 \pm n_2)\{\pm(l_1 m_2 - m_1 l_2)\,\alpha\beta\psi_{l_1,m_1,n_1}\theta_{l_2,m_2,n_2}$$
$$+ [l_1(l_1 n_2 \mp n_1 l_2)\,\alpha^2 + m_1(m_1 n_2 \mp n_1 m_2)\,\beta^2]\,\phi_{l_1,m_1,n_1}\theta_{l_2,m_2,n_2}\}, \tag{A 1b}$$

$$\mathcal{N}_{l,m,n} = im\beta \sum_{l_1+l_2=l} U_{l_1}\theta_{l_2,m,n}, \tag{A 1c}$$

while substitution of (3.6) into (3.4b) and (2.5b) yields

$$M_l = \tfrac{1}{2} \sum_{\substack{l_1+l_2=l \\ m_1+m_2=0 \\ n}} \{m_1 l_2 \alpha\beta f_{l_1,m_1,n} f_{l_2,m_2,n} - n^2 l_1 m_2 \alpha\beta g_{l_1,m_1,n} g_{l_2,m_2,n}$$
$$+ n(l_1 l_2 \alpha^2 - m_1 m_2 \beta^2)\, g_{l_1,m_1,n} f_{l_2,m_2,n}\}. \tag{A 2}$$

At lowest order (3.9a,b) and (A 1) yield

$$N^{(2)} = \tfrac{4}{3}\{2(A_1 + A_{-1}) + (\alpha^2 a_1 a^*_{-1} E_{0,2} + \beta^2 a_1 a_{-1} E_{2,0} + \text{c.c.})\}\sin 2z$$
$$- \tfrac{2}{3}\{v(a_1 E_{3,1} + a^*_{-1} E_{1,1} - a^*_1 E_{1,-1} - a_{-1} E_{3,-1}) + \text{c.c.}\}\sin z, \tag{A 3}$$

while (3.9c) and (A 2) yield

$$M^{(2)} = \tfrac{1}{2}(\lambda\mu/\alpha\beta)\, a_1 a_{-1} E_{2,0} + \text{c.c.} + \text{constant}, \tag{A 4}$$

where v, $A_{\pm 1}$ and μ are given by (3.11), (3.13d, h) respectively and c.c. denotes the complex conjugate of the expression preceding it. From (3.7) and (A 3) it follows that the order Δ terms forced by the nonlinearity have the form

$$\theta^{(2)} = \{\theta^{(2)}_{0,0,2} + (\theta^{(2)}_{0,2,2} E_{0,2} + \theta^{(2)}_{2,0,2} E_{2,0} + \text{c.c.})\} \sin 2z$$
$$+ \{(\theta^{(2)}_{3,1,1} E_{3,1} + \theta^{(2)}_{1,1,1} E_{1,1} + \theta^{(2)}_{1,-1,1} E_{1,-1} + \theta^{(2)}_{3,-1,1} E_{3,-1}) + \text{c.c.}\} \sin z. \quad \text{(A 5)}$$

Here the perturbation $\theta^{(2)}_{0,0,2} \sin 2z (= \overline{\theta^{(2)}}$ of the mean vertical temperature distribution leads to a buoyancy force, which is balanced by the vertical pressure gradient (see (3.5e)). There is no induced motion and so (3.5) simply yields

$$\theta^{(2)}_{0,0,2} = -\tfrac{2}{3}(A_1 + A_{-1}), \quad \phi^{(2)}_{0,0,2} = \psi^{(2)}_{0,0,2} = 0. \quad \text{(A 6)}$$

The remaining terms, which are calculated directly from (3.7) and (A 3), are

$$\begin{bmatrix} \phi^{(2)} \\ \psi^{(2)} \\ \theta^{(2)} \end{bmatrix}_{0,2,2} = -\gamma_1^2 \begin{bmatrix} 3 \\ 2\, 3^{\frac{1}{2}}(1+\beta^2) \\ \tfrac{4}{3}(4+\beta^2) \end{bmatrix} a_1 a^*_{-1}, \quad \begin{bmatrix} \phi^{(2)} \\ \psi^{(2)} \\ \theta^{(2)} \end{bmatrix}_{2,0,2} = -\gamma_2^2 \begin{bmatrix} 0 \\ 3^{\frac{1}{2}}/2 \\ \tfrac{1}{3} \end{bmatrix} a_1 a_{-1}, \quad \text{(A 7}a, b)$$

$$\begin{bmatrix} \phi^{(2)} \\ \psi^{(2)} \\ \theta^{(2)} \end{bmatrix}_{3,\pm 1,1} = \frac{\pm 1}{32\alpha^4} \begin{bmatrix} 3 \\ 3^{\frac{1}{2}}(3+8\alpha^2) \\ \tfrac{2}{3}(3+4\alpha^2) \end{bmatrix} v a_{\pm 1}, \quad \text{(A 7}c)$$

where
$$\gamma_1^2 = \alpha^2/(16 + 11\beta^2 + 4\beta^4), \quad \gamma_2^2 = \beta^2/(3-\beta^2). \quad \text{(A 7}d)$$

According to (3.7a) no first harmonics $E_{1,\pm 1}$ are forced at this order provided

$$\dot{\theta}^{(1)}_{1,\pm 1,1} + \Delta^{\frac{1}{2}} N^{(2)}_{1,\pm 1,1} = O(\Delta). \quad \text{(A 8)}$$

Together with (A 3) this implies that

$$\dot{a}_{\pm 1} = \pm \Delta^{\frac{1}{2}} v a^*_{\mp 1} + O(\Delta), \quad \text{(A 9}a)$$

while (3.7f) and (A 4) imply that

$$\epsilon \dot{v} = \Delta^{\frac{1}{2}} \mu a_1 a_{-1} + O(\Delta). \quad \text{(A 9}b)$$

Equations (A 9a, b) constitute a conservative system, which possesses oscillatory solutions. Since they are neither forced nor damped, any secular changes originate from the order Δ terms which have been neglected so far. At this order the only contributions to $N^{(3)}$ of interest are

$$N^{(3)}_{1,1,1} = \alpha\beta\{\psi^{(1)}_{1,-1,1}\theta^{(2)}_{0,2,2} + \psi^{(2)}_{0,2,2}\theta^{(1)}_{1,-1,1} - \psi^{(1)}_{-1,1,1}\theta^{(2)}_{2,0,2} - \psi^{(2)}_{2,0,2}\theta^{(1)}_{-1,1,1}\}$$
$$- 2\phi^{(1)}_{1,1,1}\theta^{(2)}_{0,0,2} - \alpha^2\phi^{(1)}_{1,-1,1}\theta^{(2)}_{0,2,2} - \beta^2\phi^{(1)}_{-1,1,1}\theta^{(2)}_{2,0,2} + i\beta U^{(1)}_{-2}\theta^{(2)}_{3,1,1} \quad \text{(A 10)}$$

and $N^{(3)}_{1,-1,1}$, which is given by a similar expression. Likewise the only contribution to $M^{(3)}$ of interest is

$$M^{(3)}_2 = M^{(3)}_{2+} + M^{(3)}_{2-}, \quad \text{(A 11}a)$$

where according to (A 2) and (A 7)

$$M^{(3)}_{2\pm} = (1-2\alpha^2)(f^{(2)}_{3,\pm 1,1} g^{(1)*}_{1,\pm 1,1} + g^{(2)}_{3,\pm 1,1} f^{(1)*}_{1,\pm 1,1}) \mp 2\alpha\beta(f^{(2)}_{3,\pm 1,1} f^{(1)*}_{1,\pm 1,1} - g^{(2)}_{3,\pm 1,1} g^{(1)*}_{1,\pm 1,1}). \quad \text{(A 11}b)$$

With the aid of (3.7b), (3.9), (A 6) and (A 7), the expressions (A 10) and (A 11) reduce to

$$N^{(3)}_{1,\pm1,1} = \tfrac{2}{3}\{\delta V + 2A_{\pm1} + (2+C_S \pm C_A)\,A_{\mp1}\}\,a_{\pm1}, \tag{A 12}$$

$$M^{(3)}_{2\pm} = \pm\frac{1}{2}\left(\frac{\lambda k_\pm}{\alpha\beta}\right)A_{\pm1}\,v, \tag{A 13}$$

where

$$C_S = \tfrac{1}{2}\beta^2\gamma_2^2 + 2(4+\beta^2)\,\alpha^2\gamma_1^2, \tag{A 14a}$$

$$C_A = 3^{\frac{1}{2}}\,\alpha\beta\{\gamma_2^2 - 2(5+2\beta^2)\,\gamma_1^2\} \tag{A 14b}$$

and δ and $k_\pm$ are defined by (3.13g) and (3.13i) respectively. Correct to order Δ the consistency condition for the equations (3.7) is

$$\dot{\theta}^{(1)}_{1,\pm1,1} + \Delta^{\frac{1}{2}}N^{(2)}_{1,\pm1,1} + \Delta(N^{(3)}_{1,\pm1,1} - (R^{(2)}/3^{\frac{1}{2}})\,\theta^{(1)}_{1,\pm1,1}) = 0). \tag{A 15}$$

It is used in conjunction with (A 12) to generalize (A 9a) giving (3.12a). Likewise (3.7f), (A 11a), (A 13) generalize (A 9b) giving (3.12b).

Appendix B

It is assumed that $S^{(0)}_1$ and $S^{(1)}_{-1}$ are given by (4.15). Consequently the first non-zero terms appearing in (4.19) and the average of the right-hand side of (4.6c) are

$$\mu\overline{\Phi}_- = -(C_S - C_A)\,S^{(0)}_1, \tag{B 1a}$$

$$\mu\overline{\Phi_- V} = -\tfrac{1}{2}(C_S - C_A)\,S^{(0)}_1 S^{(1)}_{-1} - \mu\delta(S^{(0)}_1 S^{(1)2}_{-1} - 4h^2)/8S^{(0)}_1, \tag{B 1b}$$

$$(\mu/\epsilon)\,\overline{\Theta} = k_+ S^{(1)}_1 - k_- S^{(1)}_{-1} + \tfrac{1}{2}(k_+ + k_-)\,S^{(1)}_{-1}, \tag{B 1c}$$

$$(\mu/\epsilon)\,\overline{\Theta V} = \tfrac{1}{2}(k_+ S^{(1)}_1 - k_- S^{(1)}_{-1})\,S^{(1)}_{-1} + (k_+ + k_-)\,(3S^{(0)}_1 S^{(1)2}_{-1} - 4h^2)/8S^{(0)}_1, \tag{B 1d}$$

$$\overline{\Phi}_+ = 0, \tag{B 1e}$$

where terms of order ϵ have been ignored. With the aid of these results it can be shown that

$$S^{(1)}_{-1}(\overline{\Phi}_+ + \overline{\Phi}_- + \epsilon^{-1}\overline{\Theta}) - 2(\epsilon^{-1}\overline{\Phi}_- S_{-1} - \overline{\Phi_- V} + \epsilon^{-1}\overline{\Theta V}) = -K(S^{(0)}_1 S^{(1)2}_{-1} - 4h^2)/4S^{(0)}_1, \tag{B 2}$$

where K is defined by (4.20b). In view of (4.19), the coefficient of h in (4.6c) can be readily calculated from (B 2). The right-hand side of (B 2) can be expressed in terms of E and S using the formulas (4.3b,d) and (4.10), so yielding (4.20a).

Appendix C

Substitution of (5.3) and (5.4) into (3.7a) shows that the forced modes satisfy the relations

$$\begin{bmatrix}\phi^{(2)}\\ \psi^{(2)}\\ \theta^{(2)}\end{bmatrix}_{\pm2L+1,1,1} = \frac{1}{3\pm2L(1\pm L)\,\alpha^2}\begin{bmatrix}3\\ 3^{\frac{1}{2}}(3\pm4L(1\pm L)\,\alpha^2)\\ \tfrac{2}{3}(3\pm2L(1\pm L)\,\alpha^2)\end{bmatrix}\vartheta_\pm, \tag{C 1}$$

where $\vartheta_\pm$ evolve according to (5.6a,b). The coefficient $M^{(3)}_{2L}$ in (5.5), which is determined by (A 2), is

$$M^{(3)}_{2L} = \mathscr{M}^{(3)}_{2L} + \mathscr{M}^{(3)*}_{-2L}, \tag{C 2a}$$

where

$$\mathscr{M}^{(3)}_{2L} = (1-(1+L)\alpha^2)(f^{(2)}_{2L+1,1,1}\, g^{(1)*}_{1,1,1} + g^{(2)}_{2L+1,1,1} f^{(1)*}_{1,1,1}) - (1+L)\,\alpha\beta(f^{(2)}_{2L+1,1,1} f^{(1)*}_{1,1,1} - g^{(2)}_{2L+1,1,1}\, g^{(1)*}_{1,1,1}). \quad \text{(C 2}b\text{)}$$

With the aid of (3.7*b*), (3.9) and (C 1), it reduces to

$$\mathscr{M}^{(3)}_{\pm 2L} = \frac{\lambda}{2L}\left(\frac{\mu\lambda_\pm}{\alpha\beta}\right)\vartheta_\pm a_1^* \quad (L > 0), \quad \text{(C 2}c\text{)}$$

where $\lambda_\pm$ is defined by (5.7*b*). Substitution of (C 2*a*, *c*) into (3.7*f*) yields (5.6*c*). Finally the contribution to $N^{(3)}$ of interest is

$$N^{(3)}_{1,1,1} = -2\phi^{(1)}_{1,1,1}\,\theta^{(2)}_{0,0,2} + i\beta\{U^{(1)*}_{2L}\,\theta^{(2)}_{2L+1,1,1} + U^{(1)}_{2L}\,\theta^{(2)}_{-2L+1,1,1}\}. \quad \text{(C 3)}$$

Upon substitution into (A 15), it yields (5.8).

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